Plan.

!!!

1. Renormalization in ϕ^4 (reminder).

1.1. Regularization and renormalization program.

Recall our **renormalization program** we considered for the ϕ^4 theory.

1.

Start with the action

$$A = \int d^4x (\frac{1}{2}(\partial\phi_0)^2 + \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4) \tag{1}$$

containing the bare field, bare mass, and bare coupling constant.

2.

Introduce some cutoff with a cutoff momentum Λ (this can be done many ways).

3.

We expect that one can give parameters m_0^2, λ_0, and the field renormalization constant Z certain dependence on Λ:

$$m_0^2 = m_0^2(\Lambda) \ , \ \lambda_0 = \lambda_0(\Lambda) \ , \ Z = Z(\Lambda) \tag{2}$$

such that the correlation functions of the renormalized field

$$\phi = Z^{-\frac{1}{2}}(\Lambda)\phi_0,$$
$$< \phi(x_1)...\phi(x_N) >= Z^{-\frac{N}{2}}(\lambda, m, \Lambda) < \phi_0(x_1)...\phi_0(x_N) > \tag{3}$$

have finite $\Lambda \to \infty$ limit.

4.

We rewrite the initial action in terms of renormalized λ, m, $\phi(x)$ introducing counterterms

$$A = \int d^4x(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\delta Z}{2}(\partial\phi)^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{4!}\phi^4) \tag{4}$$

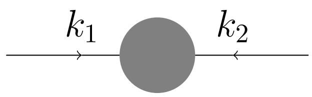
where m is an actual mass and λ is suitably defined finite coupling constant. The identity with the original action implies

$$1 + \delta Z = Z \ , \ m^2 + \delta m^2 = Z m_0^2 \ , \ \lambda + \delta\lambda = Z^2\lambda_0 \tag{5}$$

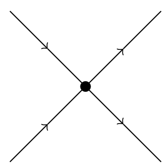
It leads to **the renormalized perturbation theory where the perturbation expansion is going by renormalized coupling constant**

λ with the following Faynman rules:

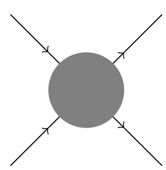
$$= \frac{1}{k^2 + m^2} \tag{6}$$



$$= (k_1^2 \delta Z - \delta m^2)(2\pi)^4 \delta(k_1 + k_2) \tag{7}$$



$$= \lambda \tag{8}$$



$$= \delta\lambda \tag{9}$$

Therefore we assume the counterterm coefficients themselves depend perturbatively (i.e. as power series) on λ:

$$\begin{aligned} \delta Z &= Z_1 \lambda + Z_2 \lambda^2 + ... \\ \delta m^2 &= b_1 \lambda + b_2 \lambda^2 ... \\ \delta\lambda &= a_1 \lambda + a_2 \lambda^2 + ... \end{aligned} \tag{10}$$

In the cutoff regularization the coefficients Z_i, b_i, a_i depends on the Λ, while in the dimensional regularization the coefficients Z_i, b_i, a_i depends on $\epsilon = 4 - d$.

1.2. Renormalization schemes.

Counterterms must cancell divergences of diagrams **but it does not fixes the counterterms completely because we have a free to change the finite parts of counterterms**. For example ϕ can be additionally renormalized by a finite value

$$\phi \to Z_{fin}^{\frac{1}{2}} \phi \tag{11}$$

One can similarly renormalize the mass m and the coupling constant λ by a finite values leading to another renormalized perturbation theory where all divergences are absorbed again. The different renormlized perturbation theories are called **renormalization schemes**.

If ϕ, m^2, λ and $\tilde{\phi}$, $\tilde{m}^2$, $\tilde{\lambda}$ are parameters in two different renormalization schemes, the corresponding proper vertices are related by

$$\Gamma^n(p_i|m^2, \lambda) = Z_{fin}(m^2, \lambda)^{-\frac{n}{2}} \Gamma^n(p_i|\tilde{m}^2(m^2, \lambda), \tilde{\lambda}(m^2, \lambda)) \tag{12}$$

These **two different renormalization schemes are two different perturbative descriptions of the same QFT. The parameters m^2, λ can be understood as a coordinates in the space of ϕ^4 QFT's**.

1.3. Physical mass and normalization of field.

The renormalization conditions define the relation between the parameters m^2, λ and ϕ with the physical values.

It is natural for example to choose parameter m^2 as a square of physical mass. Namely, we have seen that $\Gamma^2(p^2)$ becomes zero at some point $p^2 = -m^2$ so that the physical mass is given by this value of p^2.

It does not fix the normalization of the field ϕ. It is convenient to fix the field normalization demanding

$$\Gamma^2(p^2) = p^2 + m^2 + O((p^2+m^2)^2), \ when \ p^2 + m^2 \to 0 \tag{13}$$

In other words **we require the 2-point correlation function**

$$W^2(p^2) = \frac{1}{p^2+m^2} + O(1) \tag{14}$$

has pole at $p^2 = -m^2$ **and the residue at this point is equal 1**.

1.4. Coupling constant normalization.

The coupling constant has to be normalized also. It can be done by fixing the value of vertex Γ^4. The standart way to do that is to choose

$$\Gamma^4(p,p,-p,-p)|_{p^2=-m^2} = \lambda \tag{15}$$

The equations (13), (15) is one of the possibilities to choose the **Renormalization Scheme** (RS).

1.5. Renormalizable and nonrenormalizable theories.

Recall that **in renormalizable theory the divergences can be absorbed by finitely many counterterms**. So we need to fix only finite number of physical parameters to determine the theory. So they are self consistent theories.

The (perturbatively) nonrenormalizable theory has infinitely many primitive divergences which can not be obsorbed by any finite number of counterterms. Overall consistency of nonrenormalizable theories is very questionable. From purely pragmatic point of view, **the necessity to introduce infinitely many counterterms brings**

in also infinitely many free parameters, and predictive power of such theories is limited.

2. Scattering amplitudes and Green's functions (reminder).

$$S_c(\vec{p}_1, ..., \vec{p}_N|\vec{q}_1, ..., \vec{q}_M) = Z^{\frac{N+M}{2}} G_{N+M,amp}(k_1, ..., k_{N+M}) \tag{16}$$

where

$$\begin{aligned} k_i &= -p_i \ , \ i = 1, ..., N \\ k_i = q_i \ , \ i &= N+1, ..., N+M \end{aligned} \tag{17}$$

Therefore, **in order to calculate the element of S-matrix we must calculate the corresponding connected diagrams and cut off the external legs**. Here, of course, it is assumed that theory is renormalizable so that the Green's functions are finite and do not depend on the cutoff.

!!

3. Paths integral for fermionic fields.

We are now going to consider renormalizable field theories with particles of spin $\frac{1}{2}$ and 1. We statrt from $\frac{1}{2}$ spin.

3.1. Paths integral representation for Green functions of fermions.

Similarly to the scalar field we should expect the follownig relation for Dirac's field

$$\begin{aligned} < \Omega|T(\hat{\psi}_{a_1}(x_1)...\hat{\psi}_{a_N}(x_N)\hat{\bar{\psi}}_{b_1}(y_1)...\hat{\bar{\psi}}_{b_M}(y_M))|\Omega >= \\ \frac{1}{Z}\int [D\psi][D\bar{\psi}]\psi_{a_1}(x_1)...\psi_{a_N}(x_N)\bar{\psi}_{b_1}(y_1)...\bar{\psi}_{b_M}(y_M)\exp(+iS[\psi,\bar{\psi}]) \end{aligned} \tag{18}$$

where $\hat{\psi}_{a_i}(x_i)$, $\hat{\bar{\psi}}_{b_j}(y_j)$ are the Heisenberg's operators and the real-time action is

$$S = \int d^4x(\frac{\imath}{2}(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi) - m\bar{\psi}\psi) \tag{19}$$

3.2. Spin-statistics relation theorem.

It has been shown that require simultaneously

1) Lorentz invariance,

2) boundedness of the spectrum of the Hamiltonian from below,

3) causality

lead to the **Fermi-Dirac statistics for the Dirac field.**

More generally

Theorem (Pauli):

Demanding simultaneously Lorentz invariance, positivity of energy and causality, requires that integer spin particles (bosons) obey Bose-Einstein statistics, while the half-integer spin particles (fermions) obey Fermi-Dirac statistics: any multiparticle state is now anti-symmetric. For example

$$...a^{\dagger r}_{\vec{q}} a^{\dagger s}_{\vec{k}}...|0> = -...a^{\dagger s}_{\vec{k}} a^{\dagger r}_{\vec{q}}...|0> \tag{20}$$

One should expect to get the same result on the right-hand side of the expression (18). But we have classical fields on the right-hand side.

It means in fact that classical fields $\psi_a(x)$, $\bar{\psi}_b(y)$ are Grassmann valued fields.

3.3. Generating functional $Z[\eta, \bar{\eta}]$.

So we need to lern how to calculate paths integrals of fermions considering them as Grassmann variables. In particular we whant to calculate the generating functional for Dirac's fermions

$$Z[\eta,\bar{\eta}] = \int [D\psi][D\bar{\psi}] \exp[\imath \int d^4x(\bar{\psi}(\imath\gamma^\mu\partial_\mu\psi - m + \imath\epsilon)\psi + \bar{\eta}\psi + \bar{\psi}\eta)] \quad (21)$$

where $\eta(x)$ is Grassmann valued function and prove the Wick's theorem for that case.

3.4. Elements of Grassmann variables analysis.

Let us consider some properties of the Grassman algebra and functions over the Grassman numbers. Suppose we have only one grassman variable θ:

$$\theta^2 = 0 \quad (22)$$

The function $f(\theta)$ is given by Taylor expansion:

$$f(\theta) = f_0 + f_1\theta \quad (23)$$

where $f_{0,1}$ are the numbers. If we have another function $g(\theta)$ one can define the sum and product

$$\begin{aligned} f(\theta) + g(\theta) &= f_0 + g_0 + (f_1 + g_1)\theta, \\ f(\theta)g(\theta) &= f_0 g_0 + (f_0 g_1 + f_1 g_0)\theta \end{aligned} \quad (24)$$

Thus the functions $f(\theta)$ form the Grassmann algebra over the usual numbers. One can also define the derivative $\frac{\partial}{\partial\theta}$:

$$\frac{\partial}{\partial\theta} f(\theta) = f_1 \quad (25)$$

- **Grassmann algebra of functions of multiple variables:**

if we have the generators θ_i, $i = 1, ...N$ $\theta_i\theta_j = -\theta_j\theta_i$ then the general Grassmann algebra element is given by

$$f(\theta_1, ..., \theta_N) = f_0 + f_i\theta_i + f_{i_1i_2}\theta_{i_1}\theta_{i_2} + ... + f_{i_1i_2...i_N}\theta_{i_1}...\theta_{i_N} \tag{26}$$

How to calculate $\frac{\partial}{\partial\theta_k}$? Of course we want $\frac{\partial\theta_i}{\partial\theta_j} = \delta^i_j$. Let us consider

$$\begin{aligned}
\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j}(\theta_i\theta_j) = -\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j}(\theta_j\theta_i) = -\frac{\partial}{\partial\theta_i}\theta_i = -1 \\
but \\
\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j}(\theta_i\theta_j) = \epsilon\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i}(\theta_i\theta_j) = \epsilon\frac{\partial}{\partial\theta_j}\theta_j = \epsilon, \Rightarrow \\
\epsilon = -1 \Leftrightarrow \frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} = -\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i}
\end{aligned} \tag{27}$$

In general case it is convinient to formulate the derivation rule as follows. Let us introduce $\mathbb{Z}$- grading on the Grassmann algebra generated by θ_i, $i = 1, ...N$: the grading of monomial

$$|f_{i_1i_2...i_k}\theta_{i_1}\theta_{i_2}...\theta_{i_k}| = k \tag{28}$$

and

$$|\frac{\partial}{\partial\theta_i}| = -1 \tag{29}$$

Then for any monomials $A_k(\theta)$, $B_l(\theta)$ with gradings k, l

$$\begin{aligned}
A_k(\theta)B_l(\theta) = (-1)^{kl}B_l(\theta)A_k(\theta), \\
and\ by\ definition \\
\frac{\partial}{\partial\theta_i}(A_k(\theta)B_l(\theta)) = \frac{\partial A_k(\theta)}{\partial\theta_i}B_l(\theta) + (-1)^k A_k(\theta)\frac{\partial B_l(\theta)}{\partial\theta_i}
\end{aligned} \tag{30}$$

Notice that $(-1)^k$ determine $\mathbb{Z}_2$-grading on monomials.

• **The problem now what is the integral $\int f(\theta)d\theta$?**

We define the integral according to Berezin, demanding $\int f(\theta)d\theta = \int f(\theta+\eta)d\theta$:

$$\int (f_0 + f_1(\theta+\eta))d\theta = \int (f_0 + f_1\theta)d\theta + f_1\eta \int d\theta = \int (f_0 + f_1\theta)d\theta \Leftrightarrow \int d\theta = 0, \int \theta d\theta = const = 1 \Rightarrow \int f(\theta)d\theta = f_1 \tag{31}$$

Notice that

$$\int f(\theta)d\theta = \frac{\partial}{\partial\theta} f(\theta) \tag{32}$$

• **Important property of Berezin's integral.**

In case of usual (even) numbers we have:

$$x \to \lambda x \Rightarrow dx \to \lambda dx \quad \int f(x)dx \to \lambda \int f(\lambda x)dx \tag{33}$$

In order to be consistent with (31), (32) we should demand

$$\theta \to \lambda\theta \Rightarrow d\theta \to (\lambda)^{-1}d\theta,\ \frac{\partial}{\partial\theta} \to \lambda^{-1}\frac{\partial}{\partial\theta} \quad \int f(\theta)d\theta \to \lambda^{-1}\int f(\lambda\theta)d\theta \tag{34}$$

For the case of Grassman algebra generated by $\theta_1, ..., \theta_N$ the multyple integrals can be defined as repeated integrals. Because of

$$\int \theta_i d\theta_i \int \theta_j d\theta_j = \int \theta_j d\theta_j \int \theta_i d\theta_i = 1 \tag{35}$$

we obtain

$$\int \theta_i d\theta_i \theta_j d\theta_j = -\epsilon_{ij} \int \theta_j \theta_i d\theta_i d\theta_j =$$
$$-\epsilon_{ij} \int \theta_j d\theta_j = -\epsilon_{ij} = 1 \Rightarrow$$
$$\theta_i d\theta_j = -d\theta_j \theta_i \tag{36}$$

And hence

$$d\theta_i d\theta_j = -d\theta_j d\theta_i \tag{37}$$

Thus the integral over $d\theta_i$ can be defined as

$$\int d\theta_i = 0,$$
$$\int \theta_i d\theta_j = \delta_{ij} \Rightarrow$$
$$\int f(\theta_1, ..., \theta_N) d\theta_N ... d\theta_1 = f_{1...N} \tag{38}$$

- **Simplest Gaussian integral of Grassmann variables**

$$I = \int d\theta_2 d\theta_1 \exp(a\theta_1\theta_2) = \int d\theta_2 d\theta_1 (1 + a\theta_1\theta_2) = a \tag{39}$$

We can also use (34) to make a change of the variables $\theta_i = a^{-\frac{1}{2}} \eta_i$ and get

$$I = a \int d\eta_2 d\eta_1 \exp(\eta_1\eta_2) = a \tag{40}$$

- **Recall some Gaussian integrals over the usual variables:**

$$\int_{-\infty}^{\infty} \exp(-ax^2) dx = (\frac{\pi}{a})^{\frac{1}{2}}, \tag{41}$$

One can generalize this formula

$$\int \exp(-A^{ij}x_i x_j)dx_1...dx_N = (\pi)^{\frac{N}{2}}(detA)^{-\frac{1}{2}} \tag{42}$$

Recall also the complex version of the formula above

$$\int \exp(-A^{ij}\bar{z}_i z_j)dz_1 d\bar{z}_1...dz_N d\bar{z}_N = (2\pi)^N (detA)^{-1} \tag{43}$$

The mesure in the integral is invariant w.r.t. the unitary transformations $z_i \to U_i^j z_j$ because

$$\begin{aligned} dz_1...dz_N = \frac{1}{N!}\epsilon^{j_1...j_N}dz_{j_1}...dz_{j_N} \to \frac{1}{N!}\epsilon^{j_1...j_N}U_{j_1}^{i_1}...U_{j_N}^{i_N}dz_{i_1}...dz_{i_N} = \\ \frac{1}{N!}\epsilon^{j_1...j_N}U_{j_1}^{i_1}...U_{j_N}^{i_N}\epsilon_{i_1...i_N}dz_1...dz_N = \\ (detU)dz_1...dz_N \Rightarrow \\ \int \exp(-z^\dagger U^\dagger AUz)(detU)(det\bar{U})dz_1...dz_N d\bar{z}_1...d\bar{z}_N \end{aligned} \tag{44}$$

When $U^\dagger AU = diag(a_1...a_N)$ we obtain (43).

- **Grassmann algebra version of these integrals.**

  First of all

$$\int d\theta_4...d\theta_1 \exp(A_{ij}\theta_i\theta_j) = \frac{1}{2}A_{i_1i_2}A_{i_3i_4}\epsilon_{i_1i_2i_3i_4} \equiv 2Pf(A) = 2\sqrt{detA} \tag{45}$$

In a more general situation, when we have skew-symmtric $2N \times 2N$ matrix A one can use $SO(2N)$ transformation to bring this matrix to skew-diagonal form

$$\begin{pmatrix} 0 & a_1 & 0 & 0 & ... & ... \\ -a_1 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & a_2 & 0 & ... & ... \\ 0 & -a_2 & 0 & 0 & ... & ... \end{pmatrix} \tag{46}$$

Because of the measure $d\theta_1...d\theta_{2N}$ is invariant w.r.t. to this transformation $d\theta_1...d\theta_{2N} = (S^{-1})_1^{i_1}...(S^{-1})_{2N}^{i_{2N}} d\tilde{\theta}_{i_1}...\tilde{\theta}_{i_{2N}} = (detS)^{-1} d\tilde{\theta}_1...d\tilde{\theta}_{2N}$ we find

$$\int d\theta_{2N}...d\theta_1 \exp(A_{ij}\theta_i\theta_j) = 2^N a_1...a_N = 2^N Pf(A) = 2^N \sqrt{detA} \qquad (47)$$

Notice that

$$\frac{\partial}{\partial\theta} \exp(a\theta) = a,$$
$$\frac{\partial}{\partial\theta_{2N}}...\frac{\partial}{\partial\theta_1} \exp(A_{ij}\theta_i\theta_j) = Pf(A) \qquad (48)$$

- **Grassmann algebra over the complex numbers.**

First of all we define

$$\theta = \frac{\theta_1 + \imath\theta_2}{\sqrt{2}},\ \theta^* = \frac{\theta_1 - \imath\theta_2}{\sqrt{2}},$$
$$(\theta\eta)^* \equiv \eta^*\theta^* \Rightarrow$$
$$d\theta d\theta^* = \imath d\theta_2 d\theta_1 \qquad (49)$$

Then

$$\int d\theta d\theta^* \exp(-\theta^* a\theta) = a =$$
$$\frac{\partial}{\partial\theta^*}\frac{\partial}{\partial\theta} \exp(-\theta^* a\theta) \qquad (50)$$

In general case

$$\int \prod_i d\theta_i d\theta_i^* \exp(-\theta^\dagger A\theta) = detA \qquad (51)$$

where A is invertable matrix. Similar we can obtain

$$\int d\theta d\theta^* \theta\theta^* \exp(-\theta^* a\theta) = 1 = \frac{1}{a}a,$$
$$\int \prod_i d\theta_i d\theta_i^* \theta_n\theta_m^* \exp(-\theta^+ A\theta) =$$
$$\frac{\partial}{\partial A_{nm}} \int \prod_i d\theta_i d\theta_i^* \exp(-\theta^+ A\theta) = \frac{\partial det(A)}{\partial A_{nm}} =$$
$$det(A)(A^{-1})^{pq}\frac{\partial A_{pq}}{\partial A_{nm}} = det(A)(A^{-1})_{nm} \qquad (52)$$

• **Generating functional in finite dimensional case.**

$$Z[\eta,\eta^*] = \int \prod_i d\theta_i d\theta_i^* \exp(-\theta^\dagger A\theta + \eta^\dagger\theta + \theta^\dagger\eta) \tag{53}$$

One can make the change of variables:

$$\theta_i = \zeta_i + \lambda_i,\ \ \theta_i^* = \zeta_i^* + \lambda_i^*, \lambda = A^{-1}\eta \tag{54}$$

Under this change the measure is invariant so we get

$$-\theta^\dagger A\theta + \eta^\dagger\theta + \theta^\dagger\eta = -\zeta^\dagger A\zeta + \eta^\dagger A^{-1}\eta$$

$$Z[\eta,\eta^*] = \int \prod_i d\zeta_i d\zeta_i^* \exp(-\zeta^\dagger A\zeta + \eta^\dagger A^{-1}\eta) =$$

$$\exp(\eta^\dagger A^{-1}\eta) \int \prod_i d\zeta_i d\zeta_i^* \exp(-\zeta^\dagger A\zeta) = (detA)\exp(\eta^\dagger A^{-1}\eta) \tag{55}$$

3.5. $Z[\eta,\bar{\eta}]$ generating functional and correlation functions for Dirac's field.

Generalizing the formulas from (55), (53) and using the arguments which are similar to the case of scalar field we can calculate the generating functional for the Dirac's field

$$Z_D[\eta(x),\bar{\eta}(x)] = Z_D[0,0]\exp[-\int d^4x d^4y \bar{\eta} S_F(x-y)\eta] \tag{56}$$

where

$$S_F(x-y) = \imath \int \frac{d^4k}{(2\pi)^4} \frac{\exp\left[-\imath k(x-y)\right]}{k_\mu\gamma^\mu - m + \imath\epsilon} =$$
$$\imath \int \frac{d^4k}{(2\pi)^4} \frac{\exp\left[-\imath k(x-y)\right](k_\mu\gamma^\mu + m)}{k^2 - m^2 + \imath\epsilon}$$
$$\Leftrightarrow S_F(x-y) = (\imath\gamma_\mu\partial^\mu + m)D_F(x-y) \tag{57}$$

The measure of integration is coming from **Lorenz -invariant metric in the space of Grassmann-valued functions**:

$$< \delta\psi, \delta\zeta >= \int d^4x d^4y \delta\bar{\psi}(x)\delta(x-y)\delta\zeta(y) \tag{58}$$

Hence, **the integral is Lorenz-invariant**. This paths integral can be determined again as a limit of integral determined on the lattice in Euclidean space such that we obtain the integral like (53) where the number of Grassmann variables goes to infinity.

It gives Wick's theorem for Dirac's fermions:

$$< \Omega|T(\hat{\psi}_a(x_1)\hat{\bar{\psi}}_b(x_2))|\Omega >=$$
$$\frac{1}{Z_D[0,0]}(-\imath\frac{\delta}{\delta\bar{\eta}^a(x_1)})(-\imath\frac{\delta}{\delta\eta^b(x_2)})Z[\eta,\bar{\eta}]|_{\eta=\bar{\eta}=0} = S_F(x_2-x_1)_{a\bar{b}},$$
$$...... \tag{59}$$

3.6. Yukawa model.

It is helpfull to consider the application of paths integral formalism for the case of Yukawa model. This model can be considered as a simplifying version of QED where the foton is changed by a scalar field. The Lagrangian is the sum

$$L_Y = L_{KG} + L_{Dir} + L_{int} = L_{KG} + L_{Dir} - g\phi\bar{\psi}\psi \tag{60}$$

Our problem is to calculate the Green's functions by the paths integral:

$$< \Omega|T\phi(x_1)...\phi_(x_n)\psi_a(y_1)...\psi_b(y_m)\bar{\psi}_c(z_1)...\bar{\psi}_d(z_k)|\Omega >=$$
$$lim_{T\to(\infty-\imath\epsilon)}\frac{\int[D\phi][D\psi][D\bar{\psi}]\phi(x_1)...\bar{\psi}_d(z_k)\exp(\imath\int_{-T}^{T}d^4x(L_{KG}+L_{Dir}+L_{int}))}{\int[D\phi][D\psi][D\bar{\psi}]\exp(\imath\int_{-T}^{T}d^4x(L_{KG}+L_{Dir}+L_{int}))} \tag{61}$$

This theory is not free, so it is impossible to use directly the Gauss integration formulas to calculate the correlation functions. Instead we assume that coupling constant g is small and calculate the paths integral as a series over the coupling constant. Doing so we find the Feinmann rules for this theory. First of all we expand

$$\exp[\imath\int_{-T}^{T}d^4x(L_{KG}+L_{Dir}+L_{int})]=\exp[\imath\int_{-T}^{T}d^4x(L_{KG}+L_{Dir})]$$
$$(1-\imath g\int_{-T}^{T}d^4x\phi\bar{\psi}\psi+...)=$$
$$\exp[\imath\int_{-T}^{T}d^4x(L_{KG}+L_{Dir})](1-\imath\int_{-T}^{T}dtH_{int}+...) \quad (62)$$

It gives in fact the representation of interraction:

$$<\Omega|T\phi(x_1)...\phi_(x_n)\psi_a(y_1)...\psi_b(y_m)\bar{\psi}_c(z_1)...\bar{\psi}_d(z_k)|\Omega>=$$
$$lim_{T\to(\infty-\imath\epsilon)}\frac{\int[D\phi][D\psi][D\bar{\psi}]\phi(x_1)...\bar{\psi}_d(z_k)\exp[\imath\int_{-T}^{T}d^4x(L_{KG}+L_{Dir}+L_{int})]}{\int[D\phi][D\psi][D\bar{\psi}]\exp[\imath\int_{-T}^{T}d^4x(L_{KG}+L_{Dir}+L_{int})]}$$
$$=lim_{T\to(\infty-\imath\epsilon)}\frac{<\Omega|T\phi(x_1)...\bar{\psi}_d(z_k)\exp[-\imath\int_{-T}^{T}dtH_{int}]|\Omega>}{<\Omega|T\exp[-\imath\int_{-T}^{T}dtH_{int}]|\Omega>} \quad (63)$$

Hence the nominator and denominator can be represented as a Feinmann diagrams contributions with free propagators

$$D_F(x-y)\Leftrightarrow\frac{\imath}{p^2-m^2+\imath\epsilon},$$
$$S_F(x-y)\Leftrightarrow\frac{\imath(k_\mu\gamma^\mu+m)}{k^2-m^2+\imath\epsilon} \quad (64)$$

and interraction 3-legs diagram with 2 fermion lines and 1 boson line

$$-\imath g\int d^4x\Leftrightarrow-\imath g \quad (65)$$

One can consider also the correlation functions generating functional

$$
\begin{aligned}
&Z[J(x),\eta(x),\bar{\eta}(x)] = \\
&\int [D\phi][D\psi][D\bar{\psi}]\exp[\imath\int d^4x(L_{KG}+L_{Dir}-g\phi\bar{\psi}\psi+J\phi+\bar{\eta}\psi+\bar{\psi}\eta)] \\
&= \int [D\phi][D\psi][D\bar{\psi}]\exp[\imath\int d^4x(L_{KG}+L_{Dir}+J\phi+\bar{\eta}\psi+\bar{\psi}\eta)] \\
&\exp[-\imath g\int d^4y\phi\bar{\psi}\psi] = \\
&\int [D\phi][D\psi][D\bar{\psi}]\exp[\imath\int d^4x(L_{KG}+L_{Dir}+J\phi+\bar{\eta}\psi+\bar{\psi}\eta)] \\
&(1-\imath g\int d^4y\phi(y)\bar{\psi}(y)\psi(y)+...)
\end{aligned} \quad (66)
$$

The problem now is to calculate this functional using the perturbation theory over g. To do that we rewrite this functional in the form

$$
\begin{aligned}
&Z[J(x),\eta(x),\bar{\eta}(x)] = \\
&(1-\imath g\int d^4y(-\imath\frac{\delta}{\delta J(y)})(-\imath\frac{\delta}{\delta\bar{\eta}^a(y)})(-\imath\frac{\delta}{\delta\eta_a(y)})+ \\
&\frac{(-\imath g)^2}{2!}[\int d^4y(-\imath\frac{\delta}{\delta J(y)})(-\imath\frac{\delta}{\delta\bar{\eta}^a(y)})(-\imath\frac{\delta}{\delta\eta_a(y)})]^2+... \\
&\int [D\phi][D\psi][D\bar{\psi}]\exp[\imath\int d^4x(L_{KG}+L_{Dir}+J\phi+\bar{\eta}\psi+\bar{\psi}\eta)] = \\
&\exp[-\frac{\imath}{g}\int d^4y(-\imath\frac{\delta}{\delta J(y)})(-\imath\frac{\delta}{\delta\bar{\eta}^a(y)})(-\imath\frac{\delta}{\delta\eta_a(y)})] \\
&\exp[-\int d^4xd^4y(J(x)D_F(x-y)J(y)+\bar{\eta}(x)S_F(x-y)\eta(y))]
\end{aligned} \quad (67)
$$

Appendix A. Quntum mechanics paths integral in phase space, transition amplitude.

How to calculate the transition amplitude

$$
A(x_a\to x_b,T) =< x_b|\exp(-\imath\frac{HT}{\hbar})|x_a > \quad (68)
$$

It is given by the superposition of paths

$$A(x_a \to x_b, T) \equiv U(x_a, x_b; T) = \sum_{paths} \exp(\imath phase) =$$
$$\int DX(t) \exp(\imath phase) \tag{69}$$

What is phase? When $\hbar \to 0$ the stationary phase dominates, but it must be the classical trajectory

$$\frac{\delta}{\delta X(t)}(phase)|_{x_{cl}} = 0 \Rightarrow phase = S[X(t)]/\hbar \tag{70}$$

The integral can be determined by discretization of the time interval and approximazing each path $X(t)$ by the sequence of short stright segments:

$$X(t) \approx [x_a = x_0, x_1 = x(\epsilon)] \cup [x_1, x_2 = x(2\epsilon)]...$$
$$... \cup [x_{N-1} = x((N-1)\epsilon), x_N = x(N\epsilon) = x_b] \tag{71}$$

and

$$\int DX(t) = \frac{1}{C(\epsilon)} \prod_k \int_{-\infty}^{\infty} \frac{dx_k}{C(\epsilon)} \tag{72}$$

The action is

$$S = \int_0^T dt(\frac{m}{2}v^2 - V(X)) = \sum_k (\frac{m}{2}\frac{(x_{k+1} - x_k)^2}{\epsilon} - \epsilon V(\frac{x_{k+1} + x_k}{\epsilon})) \tag{73}$$

Now we consider the equation the amplitude $U(x_a, x_b; T)$ satisfy.

To this end let us whrite

$$U(x_a, x_b; T) =$$
$$\int_{-\infty}^{\infty} \frac{dx_{N-1}}{\epsilon} \exp(\frac{\imath}{\hbar}(\frac{m}{2}\frac{(x_b - x_{N-1})^2}{\epsilon} - \epsilon V(\frac{x_b + x_{N-1}}{\epsilon}))U(x_a, x_{N-1}; T - \epsilon) \tag{74}$$

and expand over $(x_{N-1} - x_b)$:

$$U(x_a, x_b; T) =$$
$$\int_{-\infty}^{\infty} \frac{dx_{N-1}}{C(\epsilon)} \exp(\frac{\imath}{\hbar}(\frac{m}{2}\frac{(x_b - x_{N-1})^2}{\epsilon}))(1 - \frac{\imath\epsilon}{\hbar}V(x_b) + ...)$$
$$(1 + (x_{N-1} - x_b)\frac{\partial}{\partial x_b} + \frac{1}{2}(x_{N-1} - x_b)^2\frac{\partial^2}{\partial x_b^2} + ...)U(x_a, x_b; T - \epsilon) \tag{75}$$

Using the formulas

$$\int dy \exp(-by^2) = \sqrt{\frac{\pi}{b}}, \int dyy \exp(-by^2) = 0, \int dyy^2 \exp(-by^2) = \frac{1}{2b}\sqrt{\frac{\pi}{b}} \quad (76)$$

we find

$$U(x_a, x_b; T) =$$

$$\frac{1}{C(\epsilon)}\sqrt{\imath\frac{2\pi\epsilon\hbar}{m}}(1 - \frac{\imath\epsilon}{\hbar}V(x_b) + \imath\frac{\epsilon\hbar}{2m}\frac{\partial^2}{\partial x_b^2})U(x_a, x_b; T-\epsilon) \quad (77)$$

In order To the limit $\epsilon \to 0$ be defined we find

$$C(\epsilon) = \sqrt{\imath\frac{2\pi\epsilon\hbar}{m}} \quad (78)$$

Hence,

$$-\imath\hbar\frac{\partial}{\partial\epsilon}(1 - \frac{\imath\epsilon}{\hbar}V(x_b) + \imath\frac{\epsilon\hbar}{2m}\frac{\partial^2}{\partial x_b^2})U(x_a, x_b; T-\epsilon) \approx$$

$$(V(x_b) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2})U(x_a, x_b, T-\epsilon) \quad (79)$$

In the limit $\epsilon \to 0$ we can write

$$\imath\hbar\frac{\partial}{\partial T}U(x_a, x_b; T) = (V(x_b) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2})U(x_a, x_b, T) \quad (80)$$

This is the Shredinger equation the evolution operator (68) satisfy

- *General case: Phase space paths integral.*

$$U(q_a, q_b; T) \equiv < q_b|\exp(-\imath TH)|q_a >=$$

$$\int\prod_{k=1} dq_k < q_b|\exp(-\imath\epsilon H)|q_{N-1} >< q_{N-1}|\exp(-\imath\epsilon H)|q_{N-2} > ...$$

$$... < q_1|\exp(-\imath\epsilon H)|q_a >$$

$$\approx \int\prod_{k=1} dq_k < q_b|(1-\imath\epsilon H)|q_{N-1} >< q_{N-1}|(1-\imath\epsilon H)|q_{N-2} > ...$$

$$... < q_1|(1-\imath\epsilon H)|q_a > \quad (81)$$

We have also

$$< q_{k+1}|f(q)|q_k >= f(q_k)\delta(q_{k+1}-q_k) = f(\frac{q_{k+1}+q_k}{2})\int\frac{dp_k}{2\pi}\exp(\imath p_k(q_{k+1}-q_k)$$

$$< q_{k+1}|f(p)|q_k >= \int\frac{dp}{2\pi} < q_{k+1}|f(p)|p >< p|q_k >=$$
$$\int\frac{dp}{2\pi}f(p)\exp(\imath p(q_{k+1}-q_k)) \quad (82)$$

If $H(p,q) = H_1(p) + H_2(q)$ we get

$$< q_{k+1}|H(p,q)|q_k >= \int\frac{dp}{2\pi}H(\frac{q_{k+1}+q_k}{2},p)\exp(\imath p(q_{k+1}-q_k), \quad (83)$$

In general one can use Weyl's ordering (operator $H(p,q)$ is a symetric function w.r.t. p and q) implying that any Hamiltonian can be represented by this way. Then

$$U(q_a,q_b;T) = \int\prod_{k=1}^{N-1}(\frac{dq_k dp}{2\pi})\exp(\imath\sum_k(p_k(q_{k+1}-q_k) - \epsilon H(\frac{q_{k+1}+q_k}{2},p_k))) \quad (84)$$

Taking the limit $\epsilon \to 0$ we find

$$U(q_a,q_b;T) = \int Dq(t)Dp(t)\exp(\imath\int_0^T dt(p\dot{q} - H(q,p))) \quad (85)$$

When $H = \frac{p^2}{2m} + V(q)$ one can take the integral over p:

$$\int\frac{dp}{2\pi}\exp(\imath(p_k(q_{k+1}-q_k) - \epsilon\frac{p_k^2}{2m}) =$$
$$\frac{1}{C(\epsilon)}\exp(\imath\frac{m}{2\epsilon}(q_{k+1}-q_k)^2) \quad (86)$$

where $C(\epsilon)$ is given by (78) so we've got the result

$$U(q_a,q_b;T) = \frac{1}{C(\epsilon)}\int\prod_{k=1}^{N-1}\frac{dq_k}{C(\epsilon)}\exp(\imath\sum_k(\frac{m(q_{k+1}-q_k)^2}{2\epsilon} - \epsilon V(\frac{q_{k+1}+q_k}{2}))) \quad (87)$$

Appendix B. Scalar field transition amplitude by paths integral.

$$H = \int d^3x (\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)) \tag{88}$$

$$< \phi_b(\vec{x})| \exp(-\imath TH)|\phi_a(\vec{x}) >=$$
$$\int D\phi D\pi \exp[\imath \int d^4x (\pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi))]$$
$$\phi_a(\vec{x}) = \phi(x^0 = 0, \vec{x}), \ \phi_b(\vec{x}) = \phi(x^0 = T, \vec{x}) \tag{89}$$

Recall that taking the integral over π we find

$$< \phi_b(\vec{x})| \exp(-\imath TH)|\phi_a(\vec{x}) >= \int D\phi \exp[\imath \int d^4x (\frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi))] \tag{90}$$

where again the integration over time dx^0 is going from $x^0 = 0$ to $x^0 = T > 0$ with the same boundary conditions for the field $\phi(x)$.

Recall how the integrals (89), (90) could be defined?

They can be defined as the limit of finite dimensional integrals. Moreover one can use the Euclidean version of these integrals first: $x^0 \to x_4 = \imath x^0$, $T \to -\imath\tau$, $(\tau > 0)$ and make an important assumption the spectrum of energy of the theory is bounded form below. In this case the transition amplitude in Euclidean space is defined as a series. Then we make an analitic continuation to the Minkowski space-time. In Euclidean space

$$< \phi_b(\vec{x})| \exp(-\tau H)|\phi_a(\vec{x}) >=$$
$$\int D\phi \exp[-\int dx_4 d^3\vec{x} (\frac{1}{2}(\frac{\partial\phi}{\partial x_4})^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi))] \tag{91}$$

Now we define the paths integral as a limit of the lattice's integral: instead of Euclidean space we consider a lattice in $\mathbb{R}^4$ with spacing Δ:

$$x_i \to x_i = \Delta n_i, \ n_i \in \mathbb{Z}, \ \phi(x_E) \to \phi_n,$$
$$\frac{\partial\phi}{\partial x_i} \to \frac{1}{\Delta}(\phi_{n+\Delta e^i} - \phi_n),$$
$$\int dx_4 d^3 \to \Delta^4 \sum_n, \ \int D\phi(x) \to \prod_n \int d\phi_n \tag{92}$$

Then the paths integral is defined by

$$\int D\phi \exp[-\int dx_4 d^3\vec{x}(\frac{1}{2}(\frac{\partial\phi}{\partial x_4})^2+\frac{1}{2}(\nabla\phi)^2+V(\phi))]=$$
$$lim_{\Delta\to 0}\prod_n\int d\phi_n \exp[-\Delta^4\sum_n L_E(\phi_n)],$$
$$L_E(\phi_n)=\frac{1}{2\Delta^2}\sum_i(\phi_{n+\Delta e^i}-\phi_n)^2+V(\phi_n) \tag{93}$$

When $V(\phi)=\frac{m^2}{2}\phi^2$ we get Gaussian integral

$$\prod_n\int d\phi_n \exp[-\frac{\Delta^2}{2}\sum_n((m^2+8)\phi_n^2-2\sum_i\phi_{n+\Delta e^i}\phi_n)]=$$
$$\prod_n\int d\phi_n \exp[-\frac{\Delta^2}{2}\sum_{n,k}\phi_k A^{k,n}\phi_n] \tag{94}$$

Appendix C. Green's functions/Correlation functions relation for scalar field.

The proof of the formula

$$< \Omega|T\phi_H(x_2)\phi_H(x_1)|\Omega >=$$
$$lim_{T\to(\infty-\imath\epsilon)}\frac{\int D\phi\phi(x_2)\phi(x_1)\exp(\imath\int_{-T}^{T}d^4xL(\phi))}{\int D\phi\exp(\imath\int_{-T}^{T}d^4xL(\phi))} \tag{95}$$

To prove this formula we consider first the following amplitude

$$< \phi_b|\exp(\imath TH)\phi_H(x_2)\phi_H(x_1)\exp(-\imath TH)|\phi_a >=$$
$$< \phi_b|\exp(-\imath(-T-x_2^0)H)\phi_S(\vec{x}_2)\exp(-\imath(x_2^0-x_1^0)H)$$
$$\phi_S(\vec{x}_1)\exp(-\imath(x_1^0+T)H)|\phi_a >=$$
$$\int D\phi(\vec{x}_1)\int D\phi(\vec{x}_2) < \phi_b|\exp(-\imath(-T-x_2^0))H)\phi_S(\vec{x}_2)|\phi_2 >$$
$$< \phi_2|\exp(-\imath(x_2^0-x_1^0)H)\phi_S(\vec{x}_1)|\phi_1 >< \phi_1|\exp(-\imath(x_1^0+T)H)|\phi_a >$$
$$=\int D\phi(\vec{x}_1)\int D\phi(\vec{x}_2)\phi(\vec{x}_1)\phi(\vec{x}_2) < \phi_b|\exp(-\imath(-T-x_2^0)H)|\phi_2 >$$
$$< \phi_2|\exp(-\imath(x_2^0-x_1^0)H)|\phi_1 >< \phi_1|\exp(-\imath(x_1^0+T)H)|\phi_a > \tag{96}$$

Notice that here $x_2^0 > x_1^0$ so the time ordering is implied. Now we consider the limit of this amplitude when $T \rightarrow (\infty - \imath\epsilon)$

$$lim_{T\rightarrow(\infty-\imath\epsilon)} \exp(-\imath TH)|\phi_a >=< \Omega|\phi_a > \exp(-\imath(\infty-\imath\epsilon)E_0)|\Omega >,$$
$$lim_{T\rightarrow(\infty-\imath\epsilon)} < \phi_b|\exp(\imath TH)| =< \Omega|\exp(\imath(\infty-\imath\epsilon)E_0) < \phi_b|\Omega >$$
$$\Longrightarrow$$
$$lim_{T\rightarrow(\infty-\imath\epsilon)} < \phi_b|\exp(\imath TH)\phi_H(x_2)\phi_H(x_1)\exp(-\imath TH)|\phi_a >=$$
$$< \phi_b|\Omega >< \Omega|\phi_a >$$
$$< \Omega|\exp(\imath(\infty-\imath\epsilon)E_0)T\phi_H(x_2)\phi_H(x_1)\exp(\imath(\infty-\imath\epsilon)E_0)|\Omega >=$$
$$< \phi_b|\Omega >< \Omega|\phi_a >< \Omega|T\phi_H(x_2)\phi_H(x_1)|\Omega > \quad (97)$$

where $|\Omega >$ is the ground state of the theory. Therefore

$$< \Omega|T\phi_H(x_2)\phi_H(x_1)|\Omega >= \frac{< \phi_b|\Omega >< \Omega|\phi_a >< \Omega|T\phi_H(x_2)\phi_H(x_1)|\Omega >}{< \phi_b|\Omega >< \Omega|\phi_a >} =$$
$$lim_{T\rightarrow(\infty-\imath\epsilon)} \frac{\int D\phi\phi(x_2)\phi(x_1)\exp(\imath\int_{-T}^{T} d^4xL(\phi))}{\int D\phi\exp(\imath\int_{-T}^{T} d^4xL(\phi))} \quad (98)$$

Appendix D. Dirac's fermions Feynman propagator.

The expression for Feynman propagator of Dirac's field is given by

$$S_{Fab}(x) \equiv< 0|T(\psi_a(x)\bar{\psi}_b(0))|0 >=$$
$$\imath\int \frac{d^4p}{(2\pi)^4}\frac{(p^\mu\gamma_\mu + m)_{ab}}{p^2 - m^2 + \imath\epsilon}\exp(-\imath px) \quad (99)$$

where

$$T(\psi_a(x)\bar{\psi}_b(y)) = \psi_a(x)\bar{\psi}_b(y),\ x^0 > y^0,$$
$$T(\psi_a(x)\bar{\psi}_b(y)) = -\bar{\psi}_b(y)\psi_a(x),\ x^0 < y^0 \quad (100)$$

It is a Green's function of Dirac wave operator

$$(\imath\gamma^\mu\partial_\mu - m)_{ab}S_{Fbc}(x) =$$
$$(\imath\gamma^\mu\partial_\mu - m)_{ab}(\imath\gamma^\mu\partial_\mu + m)_{bc}\int \frac{d^4p}{(2\pi)^4}\frac{\imath\exp(-\imath px)}{p^2 - m^2 + \imath\epsilon} =$$
$$-(\partial^\mu\partial_\mu + m^2)_{ac}D_F(x) = \imath\delta_{ab}\delta(x) \quad (101)$$

D.1. Euclidean Green function.

Recall that commutator and vacuum expectation values of KG field can be expressed in terms of certain limiting values of a single analytic function $D_C(\vec{x}, t)$. Moreover we have found that vacuum expectation values of T-ordered Heisenberg operators

$$D_F(\vec{x}, t) =< 0|T(\hat{\phi}(\vec{x}, t)\hat{\phi}(0))|0 > \tag{102}$$

can be considered also as a result of analytic continuation of the Euclidean function $D(x_E)$ satisfying the equation

$$(m^2 - \sum_{i=1}^{4}(\partial_i)^2)D(x_E) = \delta^4(x_E) \tag{103}$$

Looking at (101) it is natural to expect that similar situation takes place for the Dirac's fermions also and we can find the fermionic analog $S_{ab}(x_E)$ making the following substitutions

$$\begin{aligned} t = -\imath x_4,\ \gamma^0 = \imath\gamma_4, \\ S_F(\vec{x}, t) \to S(x_E) = (\imath\gamma_j\partial_j + m)D(x_E), \\ (\imath\gamma_j\partial_j + m)_{ab}S_{bc}(x_E) = (\imath\gamma_j\partial_j + m)^2_{ac}D(x_E) = \\ \delta_{ac}(m^2 - \sum_j \frac{\partial^2}{(\partial x_j)^2})D(x_E) = \delta_{ac}\delta(x_E) \end{aligned} \tag{104}$$

Where the Euclidean Dirac equation is implyed

$$(\imath\gamma_j\partial_j + m)\psi_E(x) = 0 \tag{105}$$

where $\{\gamma_i, \gamma_j\} = -2\delta_{ij}$. We are not going to discuss the Dirac's spinors in euclidean space.